

A Matrix Model for 2D Quantum Gravity defined by Causal Dynamical Triangulations

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Abstract

A novel continuum theory of two-dimensional quantum gravity, based on a version of Causal Dynamical Triangulations which incorporates topology change, has recently been formulated as a genuine string field theory in zero-dimensional target space (arXiv:0802.0719). Here we show that the Dyson-Schwinger equations of this string field theory are reproduced by a cubic matrix model. This matrix model also appears in the so-called Dijkgraaf-Vafa correspondence if the superpotential there is required to be renormalizable. In the spirit of this model, as well as the original large- N expansion by 't Hooft, we need no special double-scaling limit involving a fine tuning of coupling constants to obtain the continuum quantum-gravitational theory. Our result also implies a matrix model representation of the original, strictly causal quantum gravity model.

1 Introduction

Dynamical triangulations (DT) were introduced as a regularization of the Polyakov bosonic string and of two-dimensional quantum gravity [1, 2, 3]. Using this regularization, one could show that a tachyon-free version of Polyakov's bosonic string theory does not exist in target space dimensions $d > 1$ [4]. However, when viewed as a theory of 2d quantum gravity coupled to matter with central charge $c \leq 1$, the theory (non-critical string theory) did make sense. Using matrix-model techniques and other combinatorial methods, it was sometimes even advantageous to use the regularized theory for analytic calculations. Related attempts to use DT as a regularization of higher-dimensional quantum gravity [5] were less successful [6]. This triggered the introduction of Causal Dynamical Triangulations (CDT), which use causal, Lorentzian instead of Euclidean curved spacetimes as a fundamental input. Evidence has been accumulating that they provide us with a non-trivial theory of quantum gravity in four dimensions [7, 8].

While the higher-dimensional DT and CDT theories of quantum gravity at this point rely strongly on numerical simulations, the 2d CDT theory of quantum gravity can be solved analytically [9], like its 2d Euclidean DT counterpart. This is described in detail in two recent papers, where we have also developed a complete string field theory in a zero-dimensional target space for the CDT version of 2d quantum gravity [10, 11].¹ This string field theory or third quantization of 2d quantum gravity uses the formalism already developed by Ishibashi, Kawai and collaborators for the DT version of 2d quantum gravity in the context of non-critical string theory [12, 13]. For non-critical string theory, it is known from [12] that the string field theory reproduces the results of the *double-scaling limit* of the matrix models whenever the results can be compared.

Given the formal similarity between the CDT string field theory and the non-critical string field theory, it is natural to ask whether there also exists a matrix model which reproduces the results of the former. Below we will show that the answer is in the affirmative. However, since the scaling found in the CDT model is different from the conventional double-scaling limit of matrix models, a different limit needs to be taken. We will show that the limit is simply the conventional limit used in the context of the Dijkgraaf-Vafa duality to $U(N)$ supersymmetric gauge theories [14].

¹Due to the inclusion of higher-genus surfaces, this amounts to a non-trivial generalization of the original, strictly causal CDT formulation.

2 CDT string field theory

We have recently developed a string field theory for Causal Dynamical Triangulations [11]. The starting point of the CDT quantization of gravity is the assumption that in a gravitational path integral over spacetimes with a Lorentzian signature only causal geometries should be included, an idea dating back at least to [15]. How this can be done explicitly in a regularized theory, how one can rotate to Euclidean signature to perform explicit calculations, and eventually take the cut-off (or lattice spacing) to zero is described in detail in [9] for two and in [16] for three and four spacetime dimensions. We demonstrated in [10] how one can still solve the 2d model analytically when the original formulation is extended to allow the light-cone structure to become degenerate in isolated points. In [11] we generalized these results to a genuine string field theory, which enabled us in principle to calculate the amplitudes of certain spatial correlators, for two-dimensional worldsheets of any topology².

Let us briefly define the CDT string field theory, while referring to [11] for details. We will work in a Euclidean notation, which means that we started out with a Lorentzian signature, regularized the theory, rotated it to Euclidean signature as described in [9] and then took the lattice cut-off a to zero. In particular, this implies that all quantities discussed below are already *continuum* quantities.

We have a “free” Hamiltonian H_0 which describes the causal propagation of a spatial universe with respect to proper time t . Let a spatial universe with the topology of a circle of length l_2 (the “exit” loop) be separated a geodesic distance t from another spatial loop of length l_1 (the “entrance” loop), and denote the corresponding amplitude by $G_\lambda^{(0)}(l_1, l_2; t)$. It is represented by the path integral

$$G_\lambda^{(0)}(l_1, l_2; t) = \int \mathcal{D}[g_{\mu\nu}] e^{-S[g_{\mu\nu}]}, \quad (1)$$

with the (Euclidean) gravity action

$$S[g_{\mu\nu}] = \lambda \int d^2\xi \sqrt{\det g_{\mu\nu}(\xi)} + x \oint dl_1 + y \oint dl_2, \quad (2)$$

where λ is the cosmological constant, x and y are two so-called boundary cosmological constants, $g_{\mu\nu}$ is a metric representing the geometry (diffeomorphism equivalence class) $[g_{\mu\nu}]$, which is assumed to be strictly causal in the sense of the original CDT model [9]. This means essentially that the topology of its spatial sections will not change as time advances. We choose the spacetime to have the

²For earlier results in this direction we refer to [17].

topology of a cylinder, $S^1 \times [0, 1]$. In a Hilbert space language one has [9]

$$G_{\lambda}^{(0)}(l_1, l_2; t) = \langle l_2 | e^{-t H_0(l)} | l_1 \rangle, \quad H_0(l) = -l \frac{d^2}{dl^2} + \lambda l. \quad (3)$$

Next, we will generalize the class of geometries integrated over in the path integral (1). As a function of time t , spatial geometries will be allowed to branch into disconnected circles, and the resulting baby universes can subsequently merge again. Furthermore, a spatial universe will be allowed to vanish into the “vacuum” if it has length zero. These topology-changing processes can be described by the string field Hamiltonian

$$\begin{aligned} \hat{H} = & \int \frac{dl}{l} \Psi^\dagger(l) H_0(l) \Psi(l) - g \int dl_1 \int dl_2 \Psi^\dagger(l_1) \Psi^\dagger(l_2) \Psi(l_1 + l_2) \\ & - \alpha g \int dl_1 \int dl_2 \Psi^\dagger(l_1 + l_2) \Psi(l_2) \Psi(l_1) - \int \frac{dl}{l} \delta(l) \Psi(l). \end{aligned} \quad (4)$$

The operator \hat{H} is a “second quantized” Hamiltonian in the sense of many-body theory. We introduce creation and annihilation operators $\Psi^\dagger(l)$ and $\Psi(l)$ for universes of length l which act on the above-mentioned vacuum state $|0\rangle$, with defining relations

$$|l\rangle = \Psi^\dagger(l)|0\rangle, \quad \Psi(l)|l\rangle = |0\rangle, \quad \Psi(l)|0\rangle = \langle 0|\Psi^\dagger(l) = 0, \quad [\Psi(l), \Psi^\dagger(l')] = l\delta(l - l'). \quad (5)$$

In (4), g is a coupling constant of mass dimension 3, and α is a dimensionless parameter allowing us to distinguish between the merging and splitting of universes, which will be set to 1 at the end of the calculation. For $\alpha = 1$, \hat{H} is hermitian except for the presence of the tadpole term proportional to $\delta(l)$. It tells us that a universe can vanish when it has zero length, but cannot be created from nothing. Also the two interaction terms have a straightforward geometric interpretation. The first term replaces a single spatial universe of length $l_1 + l_2$ with two spatial universes of length l_1 and l_2 , while the second term represents the time-reversed process where two spatial universes merge into one, again without changing the total length $l_1 + l_2$. The coupling constant g clearly takes on the role of string coupling constant, since the splitting of spatial universes is associated with a factor g and the merging with a factor αg , making for a combined factor of αg^2 whenever the spacetime topology changes (see [11] for a detailed discussion).

We can use the string field theory associated with \hat{H} to calculate connected multi-loop correlators defined by

$$w(l_1, \dots, l_n) = \lim_{t \rightarrow \infty} \langle 0 | e^{-t \hat{H}} \Psi^\dagger(l_1) \cdots \Psi^\dagger(l_n) | 0 \rangle_{\text{connected}}. \quad (6)$$

They describe all possible ways in which an initial state of n spatial loops can

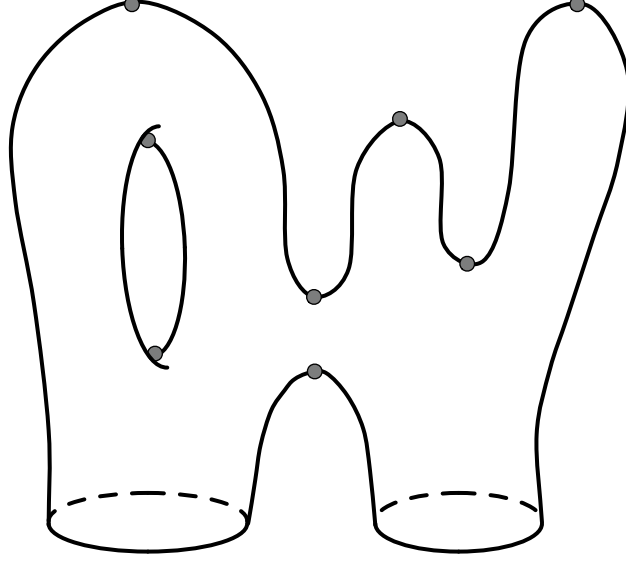


Figure 1: A typical geometry in the string field theory contributing to the amplitude $w(l_1, \dots, l_n)$ of eq. (6). Proper time progresses upwards. The dots mark singular points of the causal structure.

evolve and eventually vanish into the vacuum, while forming a connected two-dimensional geometry (c.f. Fig. 1). The amplitudes $w(l_1, \dots, l_n)$ are determined from the string field theory “partition function”

$$Z(J) = \lim_{t \rightarrow \infty} \langle 0 | e^{-t\hat{H}} e^{\int dl J(l) \Psi^\dagger(l)} | 0 \rangle \quad (7)$$

through the prescription

$$w(l_1, \dots, l_n) = \left. \frac{\delta^n F(J)}{\delta J(l_1) \cdots \delta J(l_n)} \right|_{J=0}, \quad F(J) = \log Z(J). \quad (8)$$

In [11] we derived the Dyson-Schwinger equations for the correlators $w(l_1, \dots, l_n)$. They follow from the t -independence of $Z(J)$, which leads to the relation

$$0 = \int_0^\infty dl J(l) \left\{ H_0(l) \frac{\delta F(J)}{\delta J(l)} - \delta(l) - gl \int_0^l dl' \frac{\delta^2 F(J)}{\delta J(l') \delta J(l-l')} \right. \\ \left. - gl \int_0^l dl' \frac{\delta F(J)}{\delta J(l')} \frac{\delta F(J)}{\delta J(l-l')} - \alpha gl \int_0^\infty dl' l' J(l') \frac{\delta F(J)}{\delta J(l+l')} \right\}. \quad (9)$$

One obtains the Dyson-Schwinger equations for the amplitudes $w(l_1, \dots, l_n)$ by differentiating (9) n times with respect to $J(l)$ and then setting $J(l) = 0$. The general equation at order n can be written down easily, but is involved. We will

give only the first three equations explicitly, from which the general structure should be clear. The Dyson-Schwinger equations are most conveniently formulated in terms of the Laplace-transformed amplitudes

$$w(x_1, \dots, x_n) \equiv \frac{1}{\alpha^{n-1}} \int_0^\infty dl_1 \cdots \int_0^\infty dl_n e^{-x_1 l_1 - \cdots - x_n l_n} w(l_1, \dots, l_n), \quad (10)$$

where for convenience we have rescaled the amplitudes by a factor α^{n-1} , compared with the convention used in [11]. Introducing the notation

$$V'(x) = \frac{1}{g}(\lambda - x^2), \quad V(x) = \frac{1}{g}\left(\lambda x - \frac{1}{3}x^3\right), \quad (11)$$

we obtain from (9) (see [11] for details) the equations

$$0 = \partial_x \left(-V'(x)w(x) + w^2(x) + \alpha w(x, x) \right) - \frac{1}{g}, \quad (12)$$

$$\begin{aligned} 0 = & \partial_x \left([-V'(x) + 2w(x)]w(x, y) + \alpha w(x, x, y) \right) + \\ & \partial_y \left([-V'(y) + 2w(y)]w(x, y) + \alpha w(x, y, y) \right) + \\ & + 2\partial_x \partial_y \left(\frac{w(x) - w(y)}{x - y} \right), \end{aligned} \quad (13)$$

$$\begin{aligned} 0 = & \partial_x \left([-V'(x) + 2w(x)]w(x, y, z) + \alpha w(x, x, y, z) \right) + \\ & \partial_y \left([-V'(y) + 2w(y)]w(x, y, z) + \alpha w(x, y, y, z) \right) + \\ & \partial_z \left([-V'(z) + 2w(z)]w(x, y, z) + \alpha w(x, y, z, z) \right) + \\ & 2\partial_x [w(x, y)w(x, z)] + 2\partial_y [w(x, y)w(y, z)] + 2\partial_z [w(x, z)w(y, z)] + \\ & 2 \left(\partial_x \partial_y \frac{w(x, z) - w(y, z)}{x - y} + \partial_x \partial_z \frac{w(x, y) - w(y, z)}{x - z} + \partial_y \partial_z \frac{w(x, y) - w(x, z)}{y - z} \right). \end{aligned} \quad (14)$$

Let us introduce the expansion³

$$w(x_1, \dots, x_n) = \sum_{h=0}^{\infty} \alpha^h w_h(x_1, \dots, x_n). \quad (15)$$

As shown in [11], h can be interpreted as the number of handles of the world-sheet, and the equations above can be solved iteratively in h . More precisely, the

³Note that both w and w_h are still g -dependent, although we do not write the dependence explicitly here.

equations at order α^0 allow us to determine $w_0(x)$, $w_0(x, y)$, ..., and similarly the equations at general order α^h determine $w_h(x)$, $w_h(x, y)$, etc. For example, one finds

$$w_0(x) = \frac{1}{2} \left(V'(x) + \frac{1}{g} (x - c) \sqrt{(x - c_-)(x - c_+)} \right), \quad (16)$$

$$w_0(x, y) = \frac{1}{2} \frac{1}{(x - y)^2} \left(\frac{xy - \frac{1}{2}(c_- + c_+)(x + y) + c_- c_+}{\sqrt{(x - c_-)(x - c_+)} \sqrt{(y - c_-)(y - c_+)}} - 1 \right), \quad (17)$$

where the constants c , c_{\pm} are determined by

$$c^3 - \lambda c + g = 0, \quad c_{\pm} = -c \pm \sqrt{2g/c}. \quad (18)$$

Writing the amplitudes in this fashion leads one to the surprising realization that $w_0(x)$ and $w_0(x, y)$ coincide with the large- N limit of the resolvent and the planar loop-loop correlator [18, 19, 20] of the Hermitian matrix model with potential

$$V(M) = \frac{\lambda}{g} M - \frac{1}{3g} M^3! \quad (19)$$

This is a potentially exciting result, because so far no standard formulation in terms of matrix models has been found for a CDT model, in contrast to the “old” Euclidean DT models. We will in the following section prove a more general result, which will identify the Dyson-Schwinger equations derived above with the loop equations of a Hermitian matrix model with the cubic potential (19).

3 Matrix loop equations

Let M denote an $N \times N$ Hermitian matrix, $V(M)$ a potential of the form

$$V(M) = - \sum_{k=1}^{\infty} \frac{g_k}{k} M^k, \quad (20)$$

and define the functions

$$W(x_1, \dots, x_n) = N^{n-2} \left\langle \left(\text{tr} \frac{1}{x_1 - M} \right) \cdots \left(\text{tr} \frac{1}{x_n - M} \right) \right\rangle_c. \quad (21)$$

The subscript c in $\langle O_1(M) \cdots O_n(M) \rangle_c$ denotes the connected part of the expectation value, which itself is defined as

$$\langle O_1(M) \cdots O_n(M) \rangle = \frac{\int dM O_1(M) \cdots O_n(M) e^{-N \text{tr} V(M)}}{\int dM e^{-N \text{tr} V(M)}}. \quad (22)$$

It is well known that the matrix integrals corresponding to (21) possess a large- N expansion. Assume we have the so-called one-cut solution related to this expansion. The invariance of the matrix integral under a change in variables leads to the loop equation [18, 19, 20]

$$\int_C \frac{dz}{2\pi i} \frac{V'(z)}{x-z} W(z) = W^2(x) + \frac{1}{N^2} W(x, x), \quad (23)$$

where the integration contour C encloses the cut, but not the point x . From this equation one can obtain the equations for the multi-loop correlators by differentiating with respect to the coupling constants g_k in terms of the so-called loop insertion operator [19, 21] according to

$$W(x_1, \dots, x_n) = \frac{d^{n-1}}{dV(x_2) \cdots dV(x_n)} W(x_1), \quad (24)$$

where the insertion operator is given by

$$\frac{d}{dV(x)} = \sum_{k=1}^{\infty} \frac{k}{x^{k+1}} \frac{d}{dg_k}. \quad (25)$$

For a given potential with fixed coupling constants g_k^0 one uses these relations in the following way. Assume that g_k can vary, act with the loop insertion operator on (23) as many times as needed, and then set $g_k = g_k^0$. This leads to the desired loop equations. In order to compare with the Dyson-Schwinger equations of our string field theory, we differentiate the equations obtained with respect to x , and finally find for the potential (19) the equations

$$0 = \partial_x \left(-V'(x)W(x) + W^2(x) + \frac{1}{N^2} W(x, x) \right) - \frac{1}{g}, \quad (26)$$

$$0 = \partial_x \left([-V'(x) + 2W(x)]W(x, y) + \frac{1}{N^2} W(x, x, y) \right) + \partial_x \partial_y \left(\frac{W(x) - W(y)}{x - y} \right), \quad (27)$$

$$0 = \partial_x \left([-V'(x) + 2W(x)]W(x, y, z) + \frac{1}{N^2} W(x, x, y, z) \right) + 2\partial_x \left(W(x, z)W(x, y) \right) + \partial_x \partial_y \left(\frac{W(x, z) - W(y, z)}{x - y} \right) + \partial_x \partial_z \left(\frac{W(x, y) - W(z, y)}{x - z} \right). \quad (28)$$

Using that $W(x_1, \dots, x_n)$ is a symmetric function of its arguments, we see that eqs. (26)-(28) lead to exactly the same coupled equations for W as do (12)-(14)

for w if we identify

$$\alpha = \frac{1}{N^2}. \quad (29)$$

In this case the discussion surrounding the expansion (15) is nothing but the standard discussion of the large- N expansion

$$W(x_1, \dots, x_n) = \sum_{h=0}^{\infty} \frac{1}{N^{2h}} W_h(x_1, \dots, x_n) \quad (30)$$

of the multi-loop correlators (see, for instance, [21] or the more recent papers [22, 23]). The iterative solution of these so-called loop equations is uniquely determined by $W_0(x)$ (and the assumption that $W(x_1, \dots, x_n)$ is analytic in those x_i that do not belong to the cut of the matrix model), and we have already seen that $W_0(x) = w_0(x)$.

4 Discussion and Outlook

Let us consider the matrix model corresponding to the potential (19). We can perform a simple change of variables $M \rightarrow -M - \sqrt{\lambda}$ in the matrix integral to obtain a standard matrix integral

$$Z(m, g) = \int dM e^{-NV(M)}, \quad (31)$$

where the new potential (up to an irrelevant constant term) is given by

$$V(M) = \frac{1}{g} \left(\frac{1}{2} m M^2 + \frac{1}{3} M^3 \right), \quad m = 2\sqrt{\lambda}. \quad (32)$$

It is amusing to note that the matrix integral (31) is precisely the kind of matrix integral considered in the so-called Dijkgraaf-Vafa correspondence [14], where $V(\Phi)$ is the tree-level superpotential of the adjoint chiral field Φ , which breaks the supersymmetry of the unitary gauge theory from $\mathcal{N} = 2$ to $\mathcal{N} = 1$. If one demands that this tree-level potential correspond to a renormalizable theory, its form is essentially unique, and precisely of the form (32) originally used by Dijkgraaf and Vafa, with g a dimension-three coupling constant coming from topological string theory and in the DV-correspondence related to the glueball superfield condensate in the gauge theory.

In the “old” matrix model representation of non-critical strings and 2d gravity one had to perform a fine-tuning of the coupling constants in order to obtain a continuum string or quantum gravity theory. This implemented the gluing of triangles (or, more generally, of squares, pentagons, etc.) which served as a regularization of the worldsheet. The fine-tuning of the coupling constants reflected

the fact that the link length of the triangles (the lattice spacing of the dynamical lattice) was taken to zero in the continuum limit. The situation here is different. Although CDT can be constructively defined as the continuum limit of a dynamical lattice, we have in the present work been dealing only with the associated continuum theory. Thus in our case the matrix model with the potential (19) (or (31)) already describes a *continuum* theory of 2d quantum gravity. Its coupling constants can be viewed as continuum coupling constants and the role of N is exactly as in the original context of QCD, namely, to *reorganize* the expansion in the coupling constant g . 't Hooft's large- N expansion of QCD is a reorganization of the perturbative series in the Yang-Mills coupling g_{YM} , with $1/N$ taking the role of a new expansion parameter. In this framework, after the coefficient of the term $1/N^{2h}$ of some observable has been calculated as function of the 't Hooft coupling $g_H^2 = g_{\text{YM}}^2 N$, one must take $N = 3$ for $SU(3)$, say. The situation in CDT string field theory is entirely analogous: starting from a perturbative expansion in the “string coupling constant” g (in fact, in the dimensionless coupling constant $g/\lambda^{3/2}$, as described in [10, 11]), we can reorganize it as a topological expansion in the genus of the worldsheet by introducing the expansion parameter α . For the multi-loop correlators this expansion is exactly the large- N expansion of the matrix model (19) and the coefficients, the functions $W_h(x_1, \dots, x_n)$, are exactly the multi-loop correlators for genus- h worldsheets of the CDT string field theory with $\alpha = 1$.

As a “bonus” for our treatment of generalized (and therefore slightly causality-violating) geometries, we also obtain a matrix formulation of the original two-dimensional CDT model proposed in [9], where the spatial universe was *not* allowed to split. Working out the limit as $g \rightarrow 0$ of the various expressions derived above, we see that this model corresponds to the large- N limit of the matrix model where the coupling constants go to infinity, but at the same time the cut shrinks to a point in such a way that the resolvent (or disk amplitude) survives, that is,

$$w_0(x) \rightarrow \frac{1}{x + \sqrt{\lambda}} = w_{\text{CDT}}(x). \quad (33)$$

The existence of a matrix model describing the algebraic structure of the Dyson-Schwinger equations leads automatically to the existence of Virasoro-like operators L_n , $n \geq -1$ [18, 19], which can be related to redefinitions of the time variable t in the string field theory. This line of reasoning has already been pursued by Ishibashi, Kawai and collaborators in the context of non-critical string field theory. It would be interesting to perform the same analysis in the CDT model and show that reparametrization under the change of time-variable will reappear in a natural way in the model via the operators L_n . The results should be simpler and more transparent than the corresponding results in non-critical string field theory since we have a non-trivial free Hamiltonian H_0 in the CDT model.

Acknowledgment

JA, RL, WW and SZ acknowledge support by ENRAGE (European Network on Random Geometry), a Marie Curie Research Training Network in the European Community's Sixth Framework Programme, network contract MRTN-CT-2004-005616. RL acknowledges support by the Netherlands Organisation for Scientific Research (NWO) under their VICI program.

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